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SOLVING BESSEL EQUATION OF ZERO ORDER USING WILSON WAVELETS

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Abstract. A new computational method based on Wilson wavelets is proposed for solving Bessel equation of zero order. To do this an operational matrix of integration for Wilson wavelets is obtained. Using approximation method of Wilson wavelets, Bessel equation are reduced to algebraic equations which can be solved simply to obtain an approximate solution for the problem. Several examples are presented below to demonstrate the applicability and accuracy of this method.

1. Introduction

Bessel equation is a second-order differential equation with two linearly independent solutions. The linear second-order ordinary differential equation of type

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y = 0$$

is called Bessel equation where number n is the order of the Bessel equation. The given differential equation is named after the German mathematician and astronomer Friedrich Wilhelm Bessel who studied this equation in details and showed that its solutions can be expressed in terms of a special class of functions called cylinder functions or Besel functions [4, 5]. Due to space and time constraints the interest of studying the applications of the Bessel functions will be represented as series of solution [2, 6, 7, 8, 9]. Bessel functions are series of solution to a second order differential equation that arise in diverse situations. The Bessel functions appear in many diverse scenarios, particularly the situations involving cylindrical symmetry [1, 3]. The most difficult aspect of working with the Bessel functions is determining whether it can be applied through reduction of the system of equations to Bessel differential or modified equation and then manipulating boundary conditions with appropriate application of zeroes and the coefficient values on the argument of the Bessel functions [10, 11, 12].
The special Bessel functions are widely used in solving problems of theoretical physics

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i.e, investigating wave propagation, heat conduction and vibration of membranes.

Assuming that the number n is non-integer and positive, the general solution of the Bessel equation can be written as:

$$y(x) = C_1 J_n(x) + C_2 J_{-n}(x),$$

where C_1, C_2 are arbitrary constants and $J_n(x)$, $J_{-n}(x)$ are Bessel functions of the first kind.

The Bessel functions can be represented by a series, The terms of which are expressed using the so-called Gamma function:

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(p+1)\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2p+n}.$$

The Gamma function is the generalization of the factorial function from integers to all real numbers. It has the following properties:

$$\Gamma(p+1) = p!, \quad \Gamma(p+n+1) = (n+1)(n+2)\cdots(n+p)\Gamma(n+1).$$

In this paper, we will discuss the Wilson wavelets method to solve the problems. Also, we have solved several sample examples to show the accuracy and efficiency of the Willson wavelets and compared our outcome with the exact result [14, 15, 13].

2. Wilson Wavelet

In this section, we have briefly introduced the Wilson wavelets [13] and some of its properties which are used in the sequel of the paper.
Wilson introduced a system of basis functions as follows

$$\psi_{nm}(t) = \begin{cases} \in_n \cos(2n\pi t)\omega(t - \frac{m}{2}), & m \text{ is even,} \\ \sqrt{2}\sin(2(n+1)\pi t)\omega(t - \frac{m+1}{2}), & m \text{ is odd,} \end{cases} \tag{2.1}$$

where

$$\in_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}, & n \in \mathbf{N}, \end{cases}$$

with a smooth well-localized window function ω. The elements of this system are localized around positive and negative frequency. Based on this system basis functions, Daubechies constructed an orthonormal system and called it as Wilson bases. Now we consider $\omega = \chi_{[r,r+1)}$, where $r \in \mathbf{Z}$ in (1) i.e.

$$\psi_{nm}(t) = \begin{cases} \in_n \cos(2n\pi t)\chi_{[r,r+1)}(t - \frac{m}{2}), & m \text{ is even,} \\ \sqrt{2}\sin(2(n+1)\pi t)\chi_{[r,r+1)}(t - \frac{m+1}{2}), & m \text{ is odd,} \end{cases} \tag{2.2}$$

where

$$\in_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}, & n \in \mathbf{N}. \end{cases}$$

The set $\{\psi_{nm}(t) | m \in \mathbf{Z}, n \in \mathbf{N} \cup 0\}$ is a tight frame for $L^2(\mathbf{R})$ with bound 1 [13]. Now we can show that for $r \in \mathbf{Z}$, the set $\{\psi_{nm}(t) | m \in -2r-1, -2r, n \in \mathbf{N} \cup 0\}$ in (2) is an orthonormal basis for $L^2[0,1)$, which we called as Wilson wavelets.

3. Function Approximation

Any square integrable function $f(t)$ defined on [0,1) can be expanded in terms of Wilson wavelet as [13]

$$f(t) = \sum_{n=0}^{\infty} c_{n,-2r}\psi_{n,-2r}(t) + \sum_{n=0}^{\infty} c_{n,-2r-1}\psi_{n,-2r-1}(t), \tag{3.1}$$

where $c_{nm} = \ < f(t), \psi_{nm}(t) >, \ m \in \{-2r-1, -2r\}$ and $< .,. >$ denotes the inner product on $L^2[0,1)$. If the infinite series (3.3) is truncated, then it can be written as

$$f(t) \sim \sum_{n=0}^{2^k-1} \sum_{m=-2r-1}^{-2r} c_{nm}\psi_{nm}(t) \triangleq C^T \Psi(t), \tag{3.2}$$

where C and $\Psi(t)$ are $\hat{m} = 2^{k+1}$ column vectors given by

$$C \triangleq [c_{0,-2r-1}, c_{0,-2r}, c_{1,-2r}, \cdots, c_{2^k-1,-2r-1}, c_{2^k-1,-2r}]^T,$$
$$\Psi(t) \triangleq [\psi(t)_{0,-2r-1}, \psi(t)_{0,-2r}, \cdots, \psi(t)_{2^k-1,-2r-1}, \psi(t)_{2^k-1,-2r}]^T,$$

For simplicity, (3.4) can be written as

$$f(t) \simeq P_{\hat{m}} f(t) = \sum_{i=1}^{\hat{m}} c_i \psi_i(t) \triangleq C^T \Psi(t),$$

where $c_i = c_{nm}, \ \ \psi_i(t) = \psi_{nm}(t)$, P is operational matrix of Wilson wavelets and the index i is determined by the relation $i = 2n + m + 2 + 2r$.
thus we have

$$C \triangleq [c_1, c_2, \cdots, c_{\hat{m}}]^T,$$

and

$$\Psi(t) \triangleq [\psi_1(t), \psi_2(t), \cdots, \psi_{\hat{m}}(t)]^T.$$

4. Numerical algorithm

In this section, we will obtain an algorithim for approximating the solution of second-order differential equation with given initial conditions using Wilson wavelets.

Let us consider the equation:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \tag{4.1}$$

with

$$y(0) = \alpha \ \text{ and } y'(0) = \beta.$$

where a, b, c and f are the functions of x or constants.
Now approximating a, b, c and f by using section 3. Let us assume $a, b, c, f, y'(0)$ and $y(0)$ as:

$$a(x) = A^T\psi(t), \tag{4.2}$$
$$b(x) = B^T\psi(t), \tag{4.3}$$
$$c(x) = C^T\psi(t), \tag{4.4}$$
$$f(x) = F^T\psi(t), \tag{4.5}$$
$$y'(0) = E_0^T\psi(t), \tag{4.6}$$
$$y(0) = D_0^T\psi(t), \tag{4.7}$$

In order to solve this problem with initial conditions we assume that

$$y''(x) = Y^T\psi(t), \tag{4.8}$$

Integrating (4.8) from 0 to x we get

$$y'(x) = Y^T\psi(t) + E_0^T\psi(t), \tag{4.9}$$

Further integrating (4.9) from 0 to x we get

$$y(x) = Y^T P^2\psi(t) + E_0^T P\psi(t) + D_0^T\psi(t), \tag{4.10}$$

Using (4.2) to (4.10) in the (4.1) we get

$$F\psi^T(t) = A^T\psi(t)\psi^T(t)Y + B^T\psi(t)[\psi^T(t)P^TY + \psi^T(t)E_0]+$$
$$C^T\psi(t)[\psi^T(t)P^{2T}Y + \psi^T(t)P^TE_0 + \psi^T(t)D_0], \tag{4.11}$$

By assuming $M^T\psi(t)\psi^T(t) = \psi^T(t)\tilde{M}$ [2], $A^T\psi(t)\psi^T(t)$ can be written as $\tilde{A}Y$ and using this in (4.11) we have

$$\tilde{A}Y + \tilde{B}P^TY + \tilde{B}E_0 + \tilde{C}P^{2T}Y + \tilde{C}P^TE_0^T + \tilde{C}D_0 = F, \tag{4.12}$$

Taking all Y on LHS side we get

$$\tilde{A}Y + \tilde{B}P^TY + \tilde{C}P^{2T}Y = F - (\tilde{B}E_0 + \tilde{C}P^TE_0^T + \tilde{C}D_0). \tag{4.13}$$

Now, we can easily find the Y from (4.13) and then put the values of Y in equation (4.10) to get the approximated solution of differential equation.

5. Examples

In this section, we have considered some numerical examples to illustrate the efficiency and reliability of the proposed method. These examples are considered because their exact solutions are available. In the following examples, M denote the number of vectors for Wilson wavelets.

Example 5.1. Consider the Bessel differential equation of zero order.

$$x^2y''(x) + xy'(x) + x^2y = 0,$$

with initial condition

$$y'(0) = 0 \text{ and } y(0) = 1.$$

The general solution of this equation can be expressed in terms of the so-called modified Bessel functions of the first kind as:

$$y\,(x) = C_1 J_n\,(-ix) + C_2 Y_n\,(-ix)$$

where $J_n\,(x)$ are modified Bessel function of the first kind and C_1, C_2 are arbitrary constants and $n = 0$.
The graphs are plotted in Mathematica Software.
In all the graphs given below $\tilde{y}[t]$ represent the approximated solution and $J_0(x)$ represent the series of Bessel function of zero order.
Consider the case by taking $M = 4 \times 4$ we get the approximated solution as:

$$\tilde{y}[t] = 0.23276416 - 0.00974303\cos[2\pi t] - 0.141408\sin[2\pi t] - 0.070166\sin[4\pi t].$$

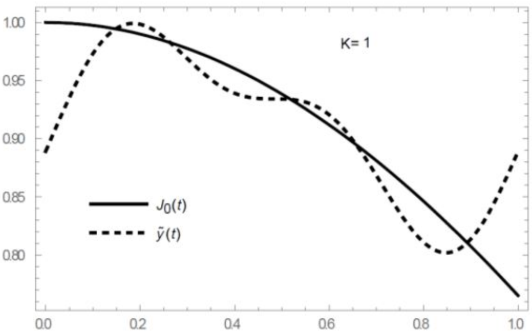

Figure 1: The graph of the exact and approximate solution for $M = 4 \times 4$.

Consider the case by taking $M = 8 \times 8$ and get the approximated solution as :
$\tilde{y}[t]$=$0.22831001807459747 + 0.01160601997286028\cos[2\pi t] +$
$0.003985314088717216\cos[4\pi t] + 0.0018360721287797189\cos[6\pi t] +$
$0.1437807423842789\sin[2\pi t] + 0.07168341886974688\sin[4\pi t] +$
$0.0473118947656462\sin[6\pi t] + 0.03556151413356036\sin[8\pi t]$.

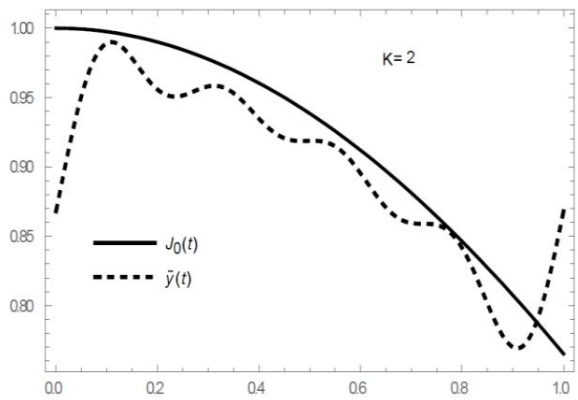

Figure 2: The graph of the exact and approximate solution for $M = 8 \times 8$.

Consider the case by taking $M = 16 \times 16$ and get the approximated solution as:
$\tilde{y}[t] = 0.234070207065769738 - 0.01887069463309948\cos[2\pi t] - 0.006487201508958887\cos[4\pi t] - 0.004069573719593767\cos[6\pi t] - 0.002855925960234084\,6\cos[8\pi t] - 0.0019501232582299324\cos[10\pi t] - 0.0011685960928342757\cos[12\pi t] - 0.0005264208072223118\cos[14\pi t] - 0.15915005940496327\sin[2\pi t] - 0.07052672059639523\sin[4\pi t] - 0.047927964774899766\sin[6\pi t] - 0.03672695580175211\,5\sin[8\pi t]$
$0.02997844293051051\sin[10\pi t] - 0.025348432337563436\sin[12\pi t] - 0.02183780599022417\sin[14\pi t] - 0.018492653001514234\sin[16\pi t]$.

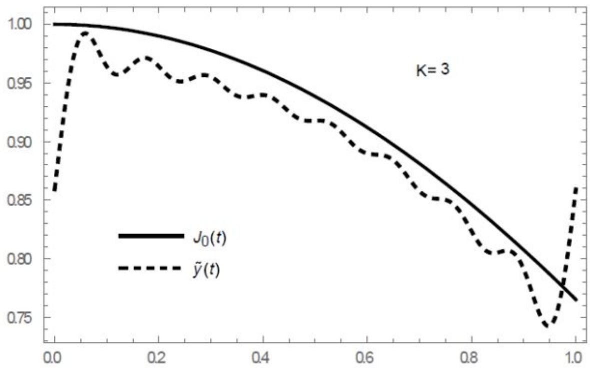

Figure 3: The graph of the exact and approximate solution for $M = 16 \times 16$.

Taking different values for M we have seen that with increase in value of M the error is much reduced. The table for L_2-error obtained by Wilson wavelets for different values of M is given below:

M	Error
4	0.034502
8	0.032589
16	0.021302

Example 5.2. Consider the differential equation

$$y''(t) + y(t) = 0,$$

with initial condition

$$y'(0) = 0 \text{ and } y(0) = 1.$$

The exact solution of the above Equation is

$$y(t) = \cos(t).$$

The graphs are plotted in Mathematica Software.
In all the graphs given below $\tilde{y}[t]$ represent the approximated solution and $y[t]$ represent the exact solution.

The approximated solution obtained in the first case by taking $M = 4 \times 4$:

$$\tilde{y}(t) \quad = \quad 4\pi \frac{(-4(\pi - 28\pi^3 + 32\pi^5)\cos[2\pi t] + (-1 + 4\pi^2)(\pi - 36\pi^3 + 64\pi^5+)}{1 - 56\pi^2 + 864\pi^4 - 1920\pi^6 + 1280\pi^8} + \frac{(2 - 56\pi^2 + 32\pi^4)\sin[2\pi t] + (1 - 24\pi^2 + 16\pi^4)\sin[4\pi t]))}{1 - 56\pi^2 + 864\pi^4 - 1920\pi^6 + 1280\pi^8}.$$

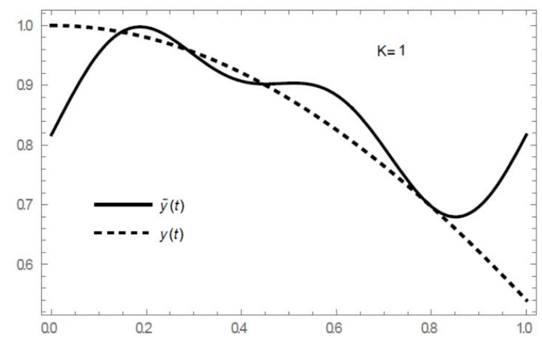

Figure 4: The graph of the exact and approximate solution for $M = 4 \times 4$.

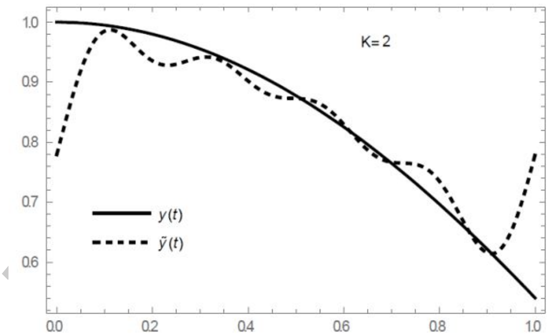

Figure 5: The graph of the exact and approximate solution for $M = 8 \times 8$.

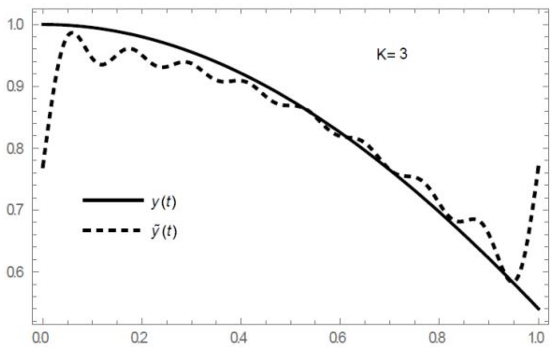

Figure 6: The graph of the exact and approximate solution for $M = 16 \times 16$.

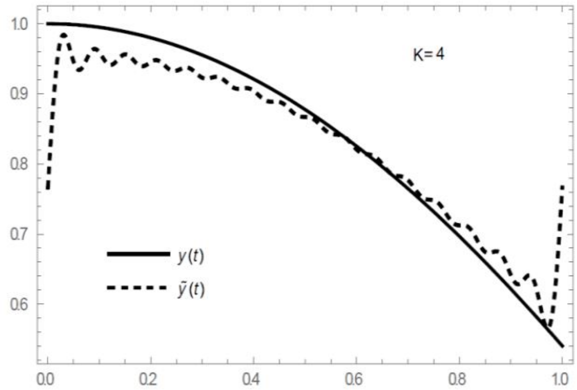

Figure 7: The graph of the exact and approximate solution for $M = 32 \times 32$.

Taking different values for M we have seen that with increase in value of M the error is much reduced. The table for L_2-error obtained by Wilson wavelets for different values of M is given below:

M	Error
4	0.0721653
8	0.0540934
16	0.0436903
32	0.0370244

6. Conclusion

In this research paper we have demonstrated that the Wilson wavelets is a powerful tool for solving Bessel equation of zero order. The proposed method is computationally efficient and the algorithm can be easily implemented on computer. The results obtained from the proposed method are compared with the exact solutions. The work done in this paper can also be extended for solving nonlinear fractional integro-differential equation. In future we will use Wilson wavelets along with collocation method to solve non linear fractional integro-differential equation and will also compare our result with Haar wavelet and CAS wavelet.

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